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FINITE BIRTH-AND-DEATH MODELS IN RANDOMLY CHANGING ENVIRONMENTS--ETC(U)
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NAVAL POSTGRADUATE SCHOOL

Monterey, California

FINITE BIRTH-AND-DEATH MODELS IN RANDOMLY CHANGING ENVIRONMENTS

by

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P. A. Jacobs
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NAVAL POSTGRADUATE SCHOOL
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This report provides computational methods for studying properties of birth-death processes whose transition rates (infinitesimal generator) varies in accordance with a random, "environmental", process. In particular, the models may be used to represent the availability of redundant systems subject to external shocks or other "common cause" failures. These models also represent search for an object when the search environment (e.g. weather) varies randomly in time.

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1. INTRODUCTION

It has long been appreciated that the simple birth-and-death Markov process often provides an adequate initial model for the behavior of service systems, populations, epidemics, and many other stochastic systems; see Feller [3] for an early classic account. Refinements, particularly in the modeling of service systems, have typically involved the replacement of exponential service times, also assumed independent, by independent random variables of "general" or arbitrary distribution, replacement of independent exponential inter-arrival times by independent "general" random variables, or both. Isolated examples in which arrival process parameters are allowed to change deterministically in time have also been studied.

Randomly appearing fluctuations in system environment, associated with weather or other change in physical surroundings, personnel changes, alteration of system usage intensity, etc., are likely to be reflected in changes in observed failure and repair rates in a system reliability context.

1.1 Examples : In order to illustrate the ideas summarized above, we introduce two specific birth and death models in random environments. Numerical results and discussion will be found in the last section.

Repairman Model. Suppose any of m machines that are in use fail independently in Markovian fashion at rate $\{\lambda_t, t > 0\}$, and, if failed, experience repair at rate $\{\mu_t, t > 0\}$. In turn, the rates $\{\lambda_t\}$ and $\{\mu_t\}$ are themselves finite-state Markov processes independent of the state of the machines, so that if n is the number of machines on repair, and j identifies the environmental state, then (n,j) is the overall state variable of the

system; conditional upon j, n changes by one unit at a time in typical birth-death fashion with parameters depending on j : the probability that an operating machine goes down in (t,t+dt) is $\lambda_j(m-n)dt + o(dt)$, while the probability that a machine on repair becomes available is $\mu_j \min(R,n)dt + o(dt)$, R being the number of repairmen. Such a setup describes groups of redundant equipments that all experience common environmental intensities simultaneously; the environmental changes are reflected in the numerical values of the failure and repair rates that prevail at any time point.

The model described also may be used to represent the behavior of a network of m timesharing computer terminals that independently send messages or programs to a common central computing facility. Suppose the rate of message transmission changes with external activity, i.e. responds to an occasional period of unusual activity, perhaps a crisis situation. In this case the "constant" demand rate λ (assumed equal for all terminals) switches almost instantaneously to a higher value, switching back to normal after the crisis elapses, but doing so repeatedly. Left alone, the central processor may well continue processing at the original rate, allowing congestion to simply build up, and eventually drop again when the demand lapses. On the other hand, remedial action may be taken. These phenomena suggest interesting and realistic questions concerning stochastic control, but these will not be considered in this paper.

Mass Search Model. Let a group of m predators attempt to round up and capture a finite group of p prey. Predators search independently and with equal intensity and effectiveness, but the detection rate is allowed to depend upon external environmental conditions that cause relatively long-term changes in, say, visibility. If a prey is detected, it is followed until lost by the predator, after which moment it is susceptible again to search, detection, and active surveillance. While predators and prey move independently, as soon as a predator begins following a particular prey the latter is removed from circulation and the remaining unattached predators continue search for the free prey.

Of interest is the long-run or stationary distribution of the number of prey under simultaneous surveillance, and also the first-passage time until a large fraction- perhaps all- of the prey are simultaneously under the eye of the predators. Here is a plausible model : the state of the system is (n,j), so that if the number of prey under surveillance, is n, and the state of the environment is j, then the probability that a free prey is detected is $\lambda_j(p-n)(m-n)dt + o(dt)$, while the probability that a prey under surveillance is lost is $\mu_j n dt + o(dt)$, where $n < \min(p,m)$ and λ_j and μ_j change in accordance with independent finite-state Markov processes, as before.

1.2 <u>The analytic structure</u> : We shall consider Markov processes $\{(X_t,Y_t),\ t > 0\}$, on the state space $\{(n,j),\ 0 < n < N,\ 1 < j < K_n\}$, with a block-tridiagonal infinitesimal generator Q :

$$Q = \begin{bmatrix} A^{(0)} & \Lambda^{(0)} & 0 & 0 & \cdots & & 0 \\ M^{(1)} & A^{(1)} & \Lambda^{(1)} & 0 & \cdots & & 0 \\ 0 & M^{(2)} & A^{(2)} & \Lambda^{(2)} & \cdots & & 0 \\ \vdots & & & & M^{(N-1)} & A^{(N-1)} & \Lambda^{(N-1)} \\ 0 & \cdots & & & 0 & M^{(N)} & A^{(N)} \end{bmatrix}. \tag{1.1}$$

Here $A^{(0)}$, $A^{(1)}$, ..., $A^{(N)}$ are square matrices, respectively of order K_0, K_1, ..., K_N. Their diagonal elements are strictly negative, all other elements are non-negative. The matrices $\Lambda^{(n)}$, $0 \leq n \leq N-1$, and $M^{(n)}$, $1 \leq n \leq N$, are rectangular, with appropriate dimensions ; their entries are non-negative. The rowsums of Q are equal to zero, therefore we have that

$$A^{(0)}\underline{e} + \Lambda^{(0)}\underline{e} = \underline{0} ,$$

$$M^{(i)}\underline{e} + A^{(i)}\underline{e} + \Lambda^{(i)}\underline{e} = \underline{0}, \qquad 1 \leq i \leq N-1,$$

$$M^{(N)}\underline{e} + A^{(N)}\underline{e} = \underline{0},$$

where $\underline{e}$ denotes a column vector with unit elements. The variable Y_t is to be interpreted as the state of the environment, and X_t as the state of the birth and death process, at time t .

We assume, furthermore, that the Markov process Q is irreducible, and we denote by <u>level n</u> the set $\{(n,j),\ 1 \leq j \leq K_n\}$, of states corresponding to the common value n for the first index. The structure (1.1) of Q permits the Markov process to move up or down by only one level at a time. It is in this respect analogous to the classical birth-and-death process, see Feller [3] or Karlin and Taylor [5].

In the examples cited above, either X or Y changes each time the Markov process undergoes a change of state. In fact, this restriction is not part of the model, and we allow X and Y to change simultaneously.

1.3 <u>Existing literature:</u> Infinite birth-and-death models in random environment have been studied for some time already; early results are found in Eisen and Tainiter [2], Purdue [11] and Yechiali [14]. More recently, Neuts [9], Chapter 6, has systematically examined a class of problems in which $N = \infty$, and Q has a special repetitive structure :

$$\Lambda^{(1)} = \Lambda^{(2)} = \ldots,$$

$$A^{(1)} = A^{(2)} = \ldots,$$

$$M^{(2)} = M^{(3)} = \ldots.$$

This structure leads to matrix-geometric stationary probability vectors and efficient algorithmic procedures.

Finite models have been examined by Torrez [12], who suggests using numerical procedures designed to solve eigenvectors for band-matrices. Hajek has considered in [4], Section 5, a finite model with repetitive structure :

$$\Lambda^{(1)} = \Lambda^{(2)} = \dots = \Lambda^{(N-2)},$$
$$A^{(1)} = A^{(2)} = \dots = A^{(N-1)},$$
$$M^{(2)} = M^{(3)} = \dots = M^{(N-1)},$$
$$K_1 = K_2 = \dots = K_{N-1} = K.$$

Hajek determines the stationary probability vector in terms of two matrices R and $\tilde{R}$, of order K, which have to be iteratively computed. Finally, Keilson et al. [6] have considered finite models with the structure (1.1) and equal K_n's. They analyse the (Laplace transform of) first passage times distributions, from which they obtain equations for moments, and for the stationary distribution. We cannot in this short space describe in detail the differences between our results and those in [4, 6, 12]; but we shall give some discussion at the end of Section 5.

This paper presents an efficient computational approach to the analysis of birth-and-death models in a Markovian environment. The emphasis is upon obtaining numerical properties of both stationary distributions (in the next section), and first-passage time (in Sections 3 and 4). The computational algorithms are discussed, and numerical examples are given, in the last two sections.

2. THE STATIONARY DISTRIBUTION

In order to determine the stationary probability distribution, and moments of first passage times, it is useful to think of the Markov process as evolving in a certain manner.

For $0 \leq n \leq N-1$, define S_n to be the restriction of the original process Q, observed during those intervals of time spent at level n, before the original process enters level n+1 for the first time, the state space of S_n is $\{(n,j),\ 1 \leq j \leq K_n\}$. Clearly all S_n, $0 \leq n \leq N-1$ are transient Markov processes. The process S_N is the restriction of the process Q to the states $\{(N,j),\ 1 \leq j \leq K_N\}$; it is an ergodic Markov process. We denote by C_n the infinitesimal generator of the process S_n, $0 \leq n \leq N$.
In order to determine the matrices C_n, we need the following result.

Lemma 1

Consider a Markov process on the state space $\{1,2,\ldots,r,r+1,r+2,\ldots,r+s\}$, with infinitesimal generator $\tilde{Q}$:

$$\tilde{Q} = \begin{bmatrix} A & \Lambda \\ 0 & 0 \end{bmatrix}, \tag{2.1}$$

where A is a square matrix of order r, Λ is a rectangular r by s matrix. The states r+1 to r+s are all absorbing.

a. The states 1 to r are all transient if and only if the matrix A is non-singular.

b. The (i,j)th entry of $(-A^{-1})$, for $1 \leq i,j \leq r$; is the expected amount of time spent in the transient state j, starting from the transient state i, before absorption in any of the absorbing states.

c. The (i,k)th entry of $(-A^{-1}\Lambda)$, for $1 \leq i \leq r$, $r+1 \leq k \leq r+s$, is the probability that, starting from the transient state i, absorption occurs in the state k.

Proof. The first assertion is proved in Neuts [9], Lemma 2.2.1. The matrix A^{-1} has all nonpositive entries. The second assertion is a consequence of Neuts and Meier [10], Corollary 2. To determine the probability $\bar{p}_{i,k}$ of being eventually absorbed in state k, $r+1 \leq k \leq r+s$, starting from state i, $1 \leq i \leq r$, we study the Markov chain embedded at instants immediately following a transition in the Markov process. The corresponding transition probability matrix $\tilde{P}$ is given by

$$\tilde{P} = \begin{bmatrix} I - \Delta^{-1}A & -\Delta^{-1}\Lambda \\ 0 & I \end{bmatrix},$$

where the matrix Δ is diagonal, with diagonal entries equal to those of A. It results from Kemeny and Snell [7], Theorem 3.3.7, p. 52, that

$$\bar{P} = [I - (I - \Delta^{-1}A)]^{-1} (-\Delta^{-1}\Lambda) = (-A^{-1})\Lambda;$$

which completes the proof of Lemma 1. □

We may now determine the matrices C_n.

Lemma 2

$$C_0 = A^{(0)}, \tag{2.2}$$

$$C_n = A^{(n)} + M^{(n)}(-C_{n-1}^{-1})\Lambda^{(n-1)}, \quad 1 \leq n \leq N. \tag{2.3}$$

Proof. The equation (2.2) is obvious :starting from level 0, the process S_0 terminates as soon as the process Q enters level 1. Meanwhile, the transitions are governed by $A^{(0)}$. Since the process Q is irreducible, the process S_0 contains only transient states, and the matrix C_0 is nonsingular, by Lemma 1. Also, $(-C_0)^{-1} \geq 0$.

We may now determine the generator C_1 of the process S_1. Let $Z_1(\tau)$, $\tau \geq 0$, be defined as follows. $Z_1(\tau) = i$ if the process S_1 is in state $(1,i)$ at time τ. Assume that $Z_1(\tau) = i$, and $Z_1(\tau + d\tau) = j \neq i$. Since $j \neq i$, a transition must have occurred in the process Q. $Z_1(\tau + d\tau) = j$ if and only if one of the following events have occured.

a. The transition is from $(1,i)$ to $(1,j)$, this happens with probability $A^{(1)}_{i,j}\, d\tau + o(d\tau)$.

b. The transition is from $(1,i)$ to $(0,k)$, for some k, <u>and</u> the process Q returns to $(1,j)$, after spending an unspecified amount of time at level 0, before visiting any other state at level 1. This happens with probability $M^{(1)}_{i,k}\, d\tau \cdot \gamma_{k,j} + o\,(d\tau)$, where $\gamma_{k,j}$ is the probability of moving from $(0,k)$ to $(1,j)$ before visiting any other state at level 1.
If we set in (2.1) $A = A^{(0)}$ and $\Lambda = \Lambda^{(0)}$, it results from Lemma 1 that $\gamma_{k,j} = [(-A^{(0)})^{-1}\Lambda^{(0)}]_{k,j}$.

It is now clear that

$$P[\,Z_1(\tau+d\tau) = j \mid Z_1(\tau) = i] = A^{(1)}_{i,j}\, d\tau + \sum_{k=1}^{K_0} M^{(1)}_{i,k}\, (-A^{(0)-1}\Lambda^{(0)})_{k,j}\, d\tau$$

$$+\, o\,(d\tau), \qquad 1 \leq i \neq j \leq K_1,$$

and similarly, that

$$P[Z_1(\tau+d\tau) = i \mid Z_1(\tau) = i] = (1+A^{(1)}_{i,i}\, d\tau) + \sum_{k=1}^{K_0} M^{(1)}_{i,k}\, (-A^{(0)-1}\Lambda^{(0)})_{k,i}\, d\tau$$

$$+\, o\,(d\tau), \qquad 1 \leq i \leq K_1 .$$

Thus, $C_1 = A^{(1)} + M^{(1)}(-A^{(0)})^{-1}\Lambda^{(0)}$,

$$= A^{(1)} + M^{(1)}(-C_0^{-1})\Lambda^{(0)},$$

which proves (2.3) for $n=1$.

Assume that (2.3) holds for n, we prove now that it holds for n+1. Let $Z_{n+1}(\tau)$, $\tau \geq 0$, be equal to i if the process S_{n+1} is in state $(n+1,i)$ at time τ.

If $Z_{n+1}(\tau) = i$, then $Z_{n+1}(\tau+d\tau) = j \neq i$ if and only if one of the following events occurs.

a. There is a transition from $(n+1,i)$ to $(n+1,j)$, with probability $A_{i,j}^{(n+1)} d\tau + o(d\tau)$.

b. There is a transition from $(n+1,i)$ to (n,k) for some k, with probability $M_{i,k}^{(n+1)} d\tau + o(d\tau)$, and the process Q returns to $(n+1,j)$, after spending an unspecified amount of time at levels $0,1,\ldots,n$, before visiting any other states at level n+1, with probability $\gamma_{k,j}^{(n)}$.

Since S_n records the visits made by Q at level n before reaching level n+1, we have by Lemma 1 that

$$\gamma_{k,j}^{(n)} = (-C_n^{-1}\Lambda^{(n)})_{k,j}. \tag{2.4}$$

It is now easy to prove that (2.3) holds for n+1, which completes the proof of the lemma.

□

[illegible] the stationary probability vector $\underline{P}$, i.e. the [illegible] $\underline{P}\,Q = \underline{0}$, $\underline{P}\,\underline{e} = 1$. We partition that vector as [illegible] where the subvectors $\underline{P}_n$ have K_n elements and [illegible] of level n, $0 \leq n \leq N$.

Theorem [illegible]

The vectors $\underline{P}_n$, [illegible] $n \leq N$, are determined by the equations

$$[illegible] \tag{2.5}$$

$$\underline{P}_{n} [illegible] M^{(n+1)} [illegible] C_n^{-1}), \qquad \text{for} \quad 0 \leq n \leq N-1, \tag{2.6}$$

$$[illegible] \tag{2.7}$$

Proof. From the structure (2.1) of Q, it results that the system $\underline{P}\,Q = \underline{0}$ may be decomposed into

$$\underline{P}_0 A [illegible] M^{(1)} = \underline{0}, \tag{2.8}$$

$$\underline{P}_{n-1} [illegible] + \underline{P}_n A^{(n)} + \underline{P}_{n+1} M^{(n+1)} = \underline{0}, \quad 1 \leq n \leq N-1, \tag{2.9}$$

$$[illegible] + \underline{P}_N A^{(N)} = \underline{0}. \tag{2.10}$$

The equations [illegible] to (2.10) are matrix equivalents of the familiar stationary equations for birth-and-death processes. They may also be [illegible] of probability balance.

From (2.9), $\underline{P}_0 = -\underline{P}_1 M^{(1)}(A^{(0)})^{-1} = \underline{P}_1 M^{(1)}(-C_0^{-1})$. Then, by recurrence, using Lemma 2, the vectors $\underline{P}_n$, $1 \leq n \leq N$, satisfy the equations (2.5) and (2.6). Since the matrix C_N is the infinitesimal generator of a finite irreducible Markov process, Equation (2.5) has a unique solution, up to a multiplicative constant, and that constant is determined by (2.7).

□

This result suggests an algorithm to compute the stationary probability vector.

Algorithm A.

A1. Determine recursively the matrices C_n, $0 \leq n \leq N$.

A2. Solve the system $\underline{\pi}_N C_N = \underline{0}$, $\underline{\pi}_N \underline{e} = 1$.

A3. Compute recursively the vectors $\underline{P}_n$, $n = N-1, \ldots, 0$, using $\underline{\pi}_N$ instead of $\underline{P}_N$.

A4. Re-normalize the vector $\underline{P}$ so obtained.

A complete analysis of this algorithm is deferred. We shall make some comments on it at the end of this section. Examples of numerical applications to certain specific models are presented at the end of the paper.

3. FIRST PASSAGE TIME TO HIGHER LEVELS

We denote by T_n the first passage time from level $n-1$ to level n, and by $\tau_{n,m}$ the first passage time from level $n-1$ to level $m \geq n$:

$$\tau_{n,m} = \inf \{t > 0 : X_t = m \mid X_0 = n-1\}, \quad 1 \leq n \leq m \leq N,$$

$$T_n = \tau_{n,n} .$$

We define, for $x \geq 0$, $1 \leq n \leq N$, $1 \leq i \leq K_{n-1}$, $1 \leq j \leq K_n$,

$$G_{i,j}^{(n)}(x) = P\,[T_n \leq x,\ Y_{T_n+0} = j \mid X_0 = n-1,\ Y_0 = i].$$

[illegible] the state of the system at time h, a simple probabilistic [illegible] shows that

$$[illegible] = (1+A_{i,i}^{(n-1)}h)g_{i,j}^{(n)}(x) + \sum_{k\neq j} A_{i,k}^{(n-1)}h\, g_{k,j}^{(n)}(x) + \Lambda_{i,j}^{(n-1)}h$$

$$+ \sum_{k=1}^{K_{n-2}} M_{i,k}^{(n-1)}h \sum_{m=1}^{K_{n-1}} g_{k,m}^{(n-1)} * g_{m,j}^{(n)}(x) + o(h), \qquad (3.1)$$

[illegible] * denotes the Stieltjes convolution. Substracting [illegible] sides of equation (3.1), dividing by h and letting [illegible] first order differential equation for $g_{i,j}^{(n)}(x)$. [illegible] the Laplace-Stieltijes transform of $g_{i,j}^{(n)}(x)$, [illegible] $[\frac{d}{dx} g_{i,j}^{(n)}(x)]\, dx$, and by $G^{(n)}(\xi)$ the matrix with entries [illegible]

[illegible] (3.1) that

$$[illegible] = A^{(n-1)}G^{(n)}(\xi) + \Lambda^{(n-1)} + M^{(n-1)}G^{(n-1)}(\xi)G^{(n)}(\xi), \qquad (3.2)$$

$$\text{for } 2 \leq n \leq N,$$

[illegible]

$$[illegible] = A^{(0)}G^{(1)}(\xi) + \Lambda^{(0)}. \qquad (3.3)$$

[illegible] be written as

$$[illegible] = D_0(\xi)\ \Lambda^{(0)}, \qquad (3.4)$$

$$[illegible] = D_{n-1}(\xi)\ \Lambda^{(n-1)}, \qquad \text{for } 2 \leq n \leq N, \qquad (3.5)$$

$$[illegible] = (\xi I - A^{(0)})^{-1}, \qquad (3.6)$$

$$[illegible] I - A^{(n)} - M^{(n)}G^{(n)}(\xi))^{-1}, \qquad 1 \leq n \leq N-1. \qquad (3.7)$$

Let $g_{i,j}^{(n,m)}(x) = P[\tau_{n,m} \leq x,\ Y_{\tau_{n,m}+0} = j \mid X_0 = n-1,\ Y_0 = i]$,

and let $G_{i,j}^{(n,m)}(\xi)$ denote the Laplace-Stieltjes transform of $g_{i,j}^{(n,m)}(x)$.
We readily obtain that

$$G^{(n,m)}(\xi) = G^{(n,m-1)}(\xi)\ G^{(m)}(\xi), \tag{3.8}$$

$$= \prod_{k=n}^{m} G^{(k)}(\xi), \tag{3.9}$$

for $1 \leq n \leq m \leq N$, where we define $G^{(n,n-1)}(\xi) = I$ for all n.

We easily prove from Lemma 2 and Equations (3.4) to (3.7) that

$$D_n\ (0) = -\ C_n^{-1}\ , \qquad \text{for } 0 \leq n \leq N-1, \tag{3.10}$$

hence $$G^{(n)}(0) = -\ C_{n-1}^{-1}\ \Lambda^{(n-1)}, \qquad \text{for } 1 \leq n \leq N, \tag{3.11}$$

which is merely another representation of Equation (2.4) , since $(G^{(n)}(0))_{i,j}$ is the probability that, starting from (n-1,i), the process Q visits (n,j) before visiting any other state at level n.

Let $U^{(n,m)} = -\dfrac{\partial}{\partial\xi}\ G^{(n,m)}(\xi)|_{\xi=0}$.

We have that

$$U_{i,j}^{(n,m)} = E[\tau_{n,m},\ Y_{\tau_{n,m}+0} = j \mid X_0 = n-1,\ Y_0 = i]\ ,$$

and $u_i^{(n,m)} = E[\tau_{n,m} | X_0 = n-1,\ Y_0 = i]$,

where $\underline{u}^{(n,m)} = U^{(n,m)}\underline{e}$.

We define $U^{(n)} = U^{(n,n)} = -\dfrac{\partial}{\partial\xi}\ G^{(n)}(\xi)|_{\xi=0}$, and $\underline{u}^{(n)} = U^{(n)}\underline{e}$.

...es of the first passage time to higher levels satisfy ...ce relations.

$$\cdot\ C_0^{-1}\ \underline{e}, \tag{3.12}$$

$$C_{n-1}^{-1}(\underline{e}+M^{(n-1)}\underline{u}^{(n-1)}), \quad \text{for } 2 \leq n \leq N, \tag{3.13}$$

$$+\ G^{(n,m-1)}(0)\ \underline{u}^{(m)}, \quad \text{for } 1 \leq n < m \leq N. \tag{3.14}$$

...y be proved by differentiating equations (3.6) and ... and evaluating the derivatives $\frac{d}{d\xi} D_n(\xi)$ at $\xi = 0$. ...rom first probabilistic principles. First, equation ...emma 1 (b) and (2.2). Then let V be the elapsed ...s enters a different level; that is, ... X_0 . By the strong Markov property, Lemma 1, and

$$\ldots,\ Y_0 = i]$$

$$n + 1 | X_0 = n\ ,\ Y_0 = i]$$

$$n - 1 | X_0 = n\ ,\ Y_0 = i]$$

$$\ldots + \sum_{k=1}^{K_{n-1}} [(-A^{(n)})^{-1}\ M^{(n)}]_{ik}\ E[T_n | X_0 = n - 1\ ,\ Y_0 = k]$$

$$\ldots M^{(n)}][(-C_{n-1}^{-1})\ A^{(n-1)}\ldots\}_{ij}\ E[T_{n+1} | X_0 = n\ ,\ Y_0 = j] \tag{3.15}$$

Hence,

$$[I - (-A^{(n)})^{-1} M^{(n)}(-C_{n-1}^{-1})\Lambda^{(n-1)}] \underline{u}^{(n+1)}$$

$$= (-A^{(n)})^{-1} e + (-A^{(n)})^{-1} M^{(n)} \underline{u}^{(n)} \tag{3.16}$$

Multiplying both sides of (3.16) by $A^{(n)}$ results in the equation

$$[A^{(n)} + M^{(n)}(-C_{n-1}^{-1})\Lambda^{n-1}] \underline{u}^{(n+1)} \tag{3.17}$$

$$= -\left[e + M^{(n)} \underline{u}^{(n)}\right].$$

Equation (3.13) now follows from (2.3) and (3.17).

Since

$$E[\tau_{n,m}|X_0 = n - 1, Y_0 = i]$$

$$= E[\tau_{n,m-1} + T_m|X_0 = n - 1, Y_0 = i]$$

$$= \underline{u}^{(n,m-1)}(i) + G^{(n,m-1)}(0) \underline{u}^{(m)} (i)$$

equation (3.14) also follows.

Any number of moments of the first passage times may be similarly obtained. We merely state the following result for second moments, without proof.

Let

$$v_i^{(n,m)} = E\,[\tau_{n,m}^2 \mid X_0 = n-1,\ Y_0 = i],$$

$$v_i^{(n)} = v_i^{(n,n)},$$

for $1 \leq n \leq m \leq N$. We have

$$\underline{v}^{(1)} = 2\,(-\,C_0^{-1})\underline{u}^{(1)}, \tag{3.18}$$

$$\underline{v}^{(n)} = 2\,(-\,C_{n-1}^{-1})\,(I + M^{(n-1)}U^{(n-1)})\underline{u}^{(n)} \tag{3.19}$$

$$+ (-\,C_{n-1}^{-1})\,M^{(n-1)}\underline{v}^{(n-1)}, \quad \text{for } 2 \leq n \leq N,$$

$$\underline{v}^{(n,m)} = \underline{v}^{(n,m-1)} + 2\,U^{(n,m-1)}\underline{u}^{(m)} + G^{(n,m-1)}(0)\,\underline{v}^{(m)}, \tag{3.20}$$

for $1 \leq n < m \leq N$.

Remarks. Theorems 1 and 2 show how the matrices C_n, $0 \leq n \leq N$, determined in Lemma 2, play a central role in the determination of both the stationary probability distribution, and the moments of first passage times to higher levels.

Example: Consider the repairman model of Section (1.1). Assume there are just two environment states, denote by $j = 1, 2$. Let the transition rate from environment state 2 to 1 be α, and from environment state 1 to 2 be β. Then $G^{(n)}(\xi)$ satisfy the following system of equations.

$$G^{(1)}_{1,j}(\xi) = \frac{\lambda_1(m)}{d_1(1)+\xi}\,\ell_j(1) + \frac{\alpha}{d_1(1)+\xi}\,G^{(1)}_{2,j}(\xi)$$

$$G^{(1)}_{2,j}(\xi) = \frac{\lambda_2(m)}{d_2(1)+\xi}\,\ell_j(2) + \frac{\beta}{d_2(1)+\xi}\,G^{(1)}_{1,j}(\xi)\,. \tag{3.21}$$

$$G^{(n+1)}_{1,j}(\xi) = \frac{\lambda_1(m-n)}{d_1(n)+\xi}\,\ell_j(1) + \frac{\mu_1[\min(R,n)]}{d_1(n)+\xi}\,[G^{(n)}(\xi)\,G^{(n+1)}(\xi)]_{1,j}$$

$$+ \frac{\alpha}{d_1(n)+\xi}\,G^{(n+1)}_{2,j}(\xi)\,. \tag{3.22}$$

$$G^{(n+1)}_{2,j}(\xi) = \frac{\lambda_2(m-n)}{d_2(n)+\xi}\,\ell_j(2) + \frac{\mu_2[\min(R,n)]}{d_2(n)+\xi}\,[G^{(n)}(\xi)\,G^{(n+1)}(\xi)]_{2,j}$$

$$+ \frac{\beta}{d_2(n)+\xi}\,G^{(n+1)}_{1,j}(\xi)$$

for $n \le m - 1$

where

$$d_1(1) = \lambda_1 m + \alpha\,;$$

$$d_2(1) = \lambda_2 m + \beta\,;$$

$$d_1(n) = \lambda_1(m-n) + \mu_1[\min(R,n)] + \alpha\,; \tag{3.23}$$

$$d_2(n) = \lambda_2(m-n) + \mu_2[\min(R,n)] + \beta\,;$$

$$\ell_j(i) = \begin{cases} 1 & \text{if } i = j\,, \\ 0 & \text{otherwise.} \end{cases}$$

The above equations can in principal be solved recursively. Finally,

$$G^{(1,n+1)}(\xi) = G^{(1)}(\xi)\, G^{(2)}(\xi) \times \ldots \times G^{(n+1)}(\xi) \; . \tag{3.24}$$

Similarly, the expected first passage times satisfy the following recursive equations.

$$E[T_1 | Y_0 = 1] = \frac{1}{d_1(1)} + \frac{\alpha}{d_1(1)} E[T_1 | Y_0 = 2]$$

$$E[T_1 | Y_0 = 2] = \frac{1}{d_2(1)} + \frac{\beta}{d_2(1)} E[T_1 | Y_0 = 1]$$

$$E[T_{n+1} | Y_0 = 1] = \frac{1}{d_1(n)} + \frac{\alpha}{d_1(n)} E[T_{n+1} | Y_0 = 2]$$

$$+ \frac{\mu_1[\min(R,n)]}{d_1(n)} \left\{ E[T_n | Y_0 = 1] + E[T_{n+1} | Y_0 = 1] \right\}$$

$$E[T_{n+1} | Y_0 = 2] = \frac{1}{d_2(n)} + \frac{\beta}{d_2(n)} E[T_{n+1} | Y_0 = 1]$$

$$+ \frac{\mu_2[\min(R,n)]}{d_2(n)} \left\{ E[T_n | Y_0 = 2] + E[T_{n+1} | Y_0 = 2] \right\}$$

4. FIRST PASSAGE TIME TO LOWER LEVELS

There is some symmetry in the process Q, which we have not exploited yet. This we proceed to do now. Instead of the processes S_n, $0 \leq n \leq N$, defined earlier, we now consider the processes $\bar{S}_n$, $0 \leq n \leq N$. For $1 \leq n \leq N$, $\bar{S}_n$ is the restriction of the process Q, observed during those intervals of time spent at level n, before the process Q moves <u>down</u> to level n-1 for the first time. All $\bar{S}_n$, $1 \leq n \leq N$, are transient Markov processes. The process $\bar{S}_0$ is the restriction of Q observed at the lowest level; it is an ergodic Markov process. We denote by $\bar{C}_n$ the infinitesimal generator of the process $\bar{S}_n$, $0 \leq n \leq N$.

We may carry out for the processes $\bar{S}_n$ exactly the same analysis as we did for the processes S_n. We indicate below the main results for two reasons.

a. The first passage times to <u>lower</u> levels must be analysed via the processes $\bar{S}_n$.
b. We shall be able to describe the precise correspondence between our analysis, and the analysis of Neuts for the <u>infinite</u> quasi-birth-and-death process.

The proof of the next lemma is omitted, since it is identical to that of Lemma 2 and Theorem 1.

Lemma 3.

The matrices $\bar{C}_n$, $0 \leq n \leq N$, are recursively determined as follows.

$$\bar{C}_N = A^{(N)} ,$$

$$\bar{C}_n = A^{(n)} + \Lambda^{(n)}(-\bar{C}_{n+1}^{-1})\, M^{(n+1)} , \qquad \text{for } 0 \leq n \leq N-1.$$

The stationary probability vector $\underline{P} = (\underline{P}_0, \underline{P}_1, \ldots, \underline{P}_N)$ is determined by the equations

$$\underline{P}_0 \bar{C}_0 = \underline{0} , \tag{4.1}$$

$$\underline{P}_n = \underline{P}_{n-1}\, \Lambda^{(n-1)}(-\bar{C}_n^{-1}), \qquad \text{for } 1 \leq n \leq N, \tag{4.2}$$

$$\sum_{n=0}^{N} \underline{P}_n\, \underline{e} = 1.$$

□

The (i,j)th entry of the matrix $\Lambda^{(n-1)}(-\bar{C}_n^{-1})$ is equal to

$$[\Lambda^{(n-1)}(-\bar{C}_n^{-1})]_{i,j} = \sum_{k=1}^{K_n} \Lambda_{i,k}^{(n-1)}(-\bar{C}_n^{-1})_{k,j} ,$$

$$= (-A_{i,i}^{(n-1)}) \sum_{k=1}^{K_n} \frac{\Lambda_{i,k}^{(n-1)}}{-A_{i,i}^{(n-1)}} (-\bar{C}_n^{-1})_{k,j} . \tag{4.3}$$

The factor $\Lambda_{i,k}^{(n-1)}/(-A_{i,i}^{(n-1)})$ is the probability that, upon leaving the state (n-1,i), the process Q moves to the state (n,k). The (k,j)th entry of $(-\bar{C}_n^{-1})$ is the expected time spent by the process in state (n,j), starting from (n,k), before hitting any state at level n-1 (by Lemma 1). Therefore, it results from (4.3) that $[\Lambda^{(n-1)}(-\bar{C}_n^{-1})]_{i,j}$ is equal to $(-A_{i,i}^{(n-1)})$ times the expected time spent in the state (n,j), before the first return to level n-1, given that the process Q starts in the state (n-1,i). This is exactly the interpretation of Neuts' matrix R for the infinite quasi-birth -and-

Let $\bar{\tau}_{m,n}$ denote the first passage time to level n, down from level m+1, for $0 \leq n \leq m \leq N-1$:

$$\bar{\tau}_{m,n} = \inf \{t>0 : X_t = n \mid X_0 = m+1\},$$

and let

$$d_i^{(m,n)} = E\,[\bar{\tau}_{m,n} \mid X_0 = m+1,\ Y_0 = i]\ ,$$

$$d_i^{(n)} = d_i^{(n,n)}\ ,$$

for $0 \leq n \leq m \leq N-1$. The proof of the next theorem is omitted, since it is identical to that of Theorem 2.

<u>Theorem 3</u>.

The expected values of the first passage time to lower levels satisfy the following recurrence relations.

$$\underline{d}^{(N-1)} = -\,\tilde{C}_N^{-1}\,\underline{e}\ , \tag{4.4}$$

$$\underline{d}^{(m)} = \underline{d}^{(m,n)} = -\,C_{m+1}^{-1}(\underline{e} + \Lambda^{(m+1)}\underline{d}^{(m+1)}),\ \text{for } 0 \leq m \leq N-2, \tag{4.5}$$

$$\underline{d}^{(m,n)} = \underline{d}^{(m,n+1)} + \tilde{G}^{(m,n+1)}(0)\underline{d}^{(n)}\ ,\ \text{for } 0 \leq n < m \leq N-1, \tag{4.6}$$

where

$$\tilde{G}^{(m,n)}(0) = \prod_{k=m}^{n} (-C_{k+1})\ M^{(k+1)},\ \text{for } 0 \leq n \leq m \leq N-1, \tag{4.7}$$

□

Observe that in the right-hand side of (4.7) , the left most matrix in the product is $(-\tilde{C}_{m+1}^{-1})$, not $(-\,\tilde{C}_{n+1}^{-1})$.

5. REMARKS ABOUT THE COMPUTATIONAL ALGORITHMS

The algorithm A described in Section 2 is numerically stable under a large range of values for the entries of the generator Q of (1.1). The matrices $(-C_n^{-1})$ have only non-negative entries. Moreover, since they measure the time spent at level n only, before moving to level n+1, they usually are of the same order of magnitude for each n, even for large values of N, if the entries of the matrices $\Lambda^{(n)}$ are of the same order of magnitude for each n, and similarly for the matrices $M^{(n)}$. A potential source of trouble exists when the Λ's are either very small or very large, compared to the other elements of Q. In that case, the expected times spent at one level before moving up are respectively very large or very small, and there is a risk of encountering overflow or underflow problems, when determining the matrices $(-C_n^{-1})$.

The steps A3 and A4 of Algorithm A are more delicate. We start step A3 with a vector $\underline{\pi}_N$ normalized by $\underline{\pi}_N \, \underline{e} = 1$. If N is very large, it is likely that the vector $\underline{P}_N$ will be much smaller than $\underline{\pi}_N$, and there is a real risk of running into overflow problems while performing step A3. In order to overcome this difficulty, we have merged the two steps A3 and A4, and re-normalized the vectors each time a new subvector is determined (see Algorithm B in Appendix A).

It results from Theorem 2 that one may compute the vectors $\underline{u}^{(n,m)}$ at the same time as one is preparing the evaluation of the stationary probability distributions, i.e. during step A1 of Algorithm A. In the numerical examples presented in the next section, we have determined the first passage times from level 0 to level m, for $1 \leqslant m \leqslant N$. The corresponding algorithm

Finally, we compare the numerical efficiency of three approaches. As observed by Torrez [12], the matrix Q is a band-matrix, and there exist efficient numerical procedures to solve eigenvectors for band-matrices. The complexity of such procedures, for the matrix Q, is $O(N, \tilde{K}^3)$, where $\tilde{K} = \max\{K_n, 0 < n < N\}$ (Wilkinson and Reinsh [13], p. 70). The crucial step in our algorithm resides in the inversion of the matrices C_n, $0 < n < N-1$, and the solution of the system (2.5). The corresponding complexity is $O(\sum_{n=0}^{N} K_n^3)$. In Keilson et al. [6], 2 N matrices have to be inverted, and N systems have to be solved. Again, the complexity is $O(\sum_{n=0}^{N} K_n^3)$. Clearly, the three approaches have globally similar numerical efficiency; in order to distinguish among them, one would have to determine the coefficients implicit in the O(.) notation. However, it must be observed that the last two have the additional advantage of offering clear probabilistic interpretations of the computed quantities.

6. NUMERICAL RESULTS

6.1 Machine repairman model. We consider a system with N=5 machines and R=1 repairman. The system can be in K=2 environments. The system remains in the environment state j (j = 1,2) for an exponentially distributed random interval of time, with parameter α_j. The failure rate of each machine is equal to λ_j in the j^{th} environment, with $\lambda_1 = 0.12$, and $\lambda_2 = 0.06$. The repair rate of the repairman is $\mu = 1$ in both environments.

Our objective is to measure how the rate of changes in the environment influences the system behaviour. In order to do so, we set $\alpha_1 = \beta/2$, $\alpha_2 = \beta$, and chose different values for β. Then the stationary probabilities of being in each environment remain constant and are given by $\gamma_1 = 2/3$ and $\gamma_2 = 1/3$. We expect that if β is large, then the environment changes rapidly, and the system is only influenced by the average failure rate $\lambda_0 = \gamma_1\lambda_1 + \gamma_2\lambda_2 = 0.1$.

On the other hand, if β is small, then the environment stays for long periods of time in the same state, and this should affect the dynamic behaviour of the system.

We denote by $\xi_i(\beta)$, $0 \leq i \leq 5$, the marginal distribution of the number of machines on repair, for a given value of β.
We furthermore denote by $\eta_i(\lambda)$, $0 \leq i \leq 5$, the probability distribution for the classical machine repairman system, with constant failure rate λ.

In table I, we give the cumulative probabilities, corresponding to $\underline{\eta}(\lambda)$, for $\lambda = \lambda_0$, λ_1 and λ_2, and to $\underline{\xi}(\beta)$, for $\beta = 10^{-5}$, 10^{-2}, 10^{-1}, 1, 10, 10^2, 10^5. We observe that the distributions of all the systems in a random environment are close to the distribution for the system with a unique, average rate λ_0, with increasing differences when the environment changes more slowly. We also observe that $\underline{\xi}(10^{-5}) = \gamma_1\underline{\eta}(\lambda_1) + \gamma_2\underline{\eta}(\lambda_2)$, up to five decimal places.

We conclude therefore that for the present model, the random environment has little effect on the marginal stationary distribution of the number of machines on repair.

In order to measure the influence of the random environment on the dynamic behaviour of the system, we have computed the average time needed to reach the states [n machines on repair], $1 \leq n \leq 5$, starting from the state [0 machine on repair]. We present on Figure 1 the results for n=5, which is interpretable as the "time to complete failure". The functions are as follows :

- 1 and 2 : $f_1(\beta)$ and $f_2(\beta)$, where $f_j(\beta)$ = E{time to reach the state [all machines on repair] starting from the state [0 machine on repair and environment j]}.
- 3, 4 and 5 : $g(\lambda_1)$, $g(\lambda_2)$ and $g(\lambda_0)$, where $g(\lambda)$ = E{time to reach the state [all machines on repair], starting from [0 machine on repair]},for a system with constant failure rate λ.

- 6 : $\gamma_1 f_1(\beta) + \gamma_2 f_2(\beta)$, equal to the <u>stationary</u> expected time to complete failure, starting from [0 machine on repair].

We clearly observe several typical ranges of values for β. If $\beta > 10$, the environment changes so rapidly that the expected time to complete failure is equal to the time for a system with unique failure rate equal to λ_0. For $10^{-1} < \beta < 10$, the expected time to complete failure does not depend on the initial environment state. For $\beta < 10^{-6}$, the environment changes so slowly that complete failure is reached before a change of environment occurs. In fact, it seems that the system is almost completely decomposed in two different systems, one corresponding to each environment, with very slow migrations from one to the other. Also, we note that for $\beta < 10^{-3}$, the stationary expected time to complete failure looses any practical significance. Finally, the average failure rate λ_0 yields an <u>overestimation</u> of the time to complete failure for $10^{-3} < \beta < 1$, that is, for values of β which are neither much lower, nor much higher than the failure rates.

6.2 <u>Mass search model</u>. We define a reference model with p=15 prey and m predators. The system can be in K=2 environments, with parameters α_1 and $\alpha_2 = 2\,\alpha_1$. The detection and loss rates are respectively given by $\lambda_1 = .001$, $\lambda_2 = .005$, $\mu_1 = .02$ and $\mu_2 = .01$. Thus, environment 2 is more favorable to the predators, since the detection rate is higher, and the loss rate is lower; however, environment 1 lasts on the average longer than environment 2.

For this model, the rate of changes in the environment has an influence on the marginal stationary distribution of the number of prey under surveillance. In Table II, we give that distribution for five different models. Columns 1 and 2 correspond to models with a unique environment (respectively environments 1 and 2).

Columns 3 and 4 correspond to the reference model, respectively with $\alpha_1 = 10^{-4}$ (slow environment changes), and $\alpha_1 = 1$ (fast environment changes). Column 5 corresponds to a model with a unique, average environment : $\lambda = \frac{2}{3}\lambda_1 + \frac{1}{3}\lambda_2$, $\mu = \frac{2}{3}\mu_1 + \frac{1}{3}\mu_2$. Note that the distribution in column 3 has two modes, respectively corresponding to the modes in columns 1 and 2.

We have also measured how the system responds to increases in the efficiency of the predators. We have considered two ways for the predators to become more efficient. The first one is by becoming more numerous, the second by increasing the probability of detecting a free prey, or by decreasing the probability of losing a prey under surveillance. We present on Figures 2 and 3 the values of $t_{n,j}$, n=5,10,15, j=1,2, where

$$t_{n,j} = E\,[\text{time until n prey are under surveillance} \mid \text{at time 0, 0 prey under surveillance, environment state is j}],$$

for the reference model with $\alpha_1 = .0001$, and m equals 15 to 45. It clearly appears that environment 2 is more favorable for the predators. The curve for $t_{15,1}$ presents a plateau which we explain as follows. We denote by $T'_{n,j}$ the time until n prey are under surveillance given that initially zero prey are under surveillance, for a system with a unique environment, identical to environment j. For values of m less than 32, say, $T'_{15,1}$ is so large (greater than 10^5 on the average) that the reference model with $\alpha_1 = .0001$ switches to environment 2 before 15 prey are under surveillance (the switch occurs on the average after 10^4 units of time). Once the model is in environment 2, it takes on the average 10^2 to 10^3 units of time to have all preys under surveillance. This occurs before the environment switches back to 1, and the total elapsed time is, on the average, approximately 10^4.

On Figures 4 to 7, we present the values of $t_{n,j}$, n=5,10,15, j=1,2, for models derived from the reference model. We consider α_1 = .0001, m equals to 15 and 20, and we multiply the rates λ_j by $\sqrt{\gamma}$, we divide the rates μ_j by $\sqrt{\gamma}$, with $\gamma > 1$, so that the probability ratio's "probability of detecting / probability of losing" are uniformly multiplied by γ in each state. These figures may be used in conjunction with Figures 2 and 3 to measure trade-off such as the following one. Suppose we start from the reference model with m=15, and that we double the number of predators to m=30. This will entail a reduction on $t_{15,1}$, which can be measured on Figure 2. We can then measure on Figure 4 that the probability ratio must be multiplied by a factor $\gamma \approx 3$ in order to obtain the same reduction.

In Table III, we give the approximate values of γ which give the same reduction as doubling the number of predators, for m=15 and m=20.

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APPENDIX A.

Algorithm B : evaluation of the stationary probability distribution, the first and second moments of the time to reach level m, $1 \leqslant m \leqslant N$, starting from level 0.

B1 First passage times from level 0 to higher levels.

B1.1 Initial values.

$$\{\ (-C_0^{-1}) \leftarrow -(A^{(0)})^{-1}\ ; \qquad \text{(Equation (2.2))}$$

$$\underline{u}^{(1)} \leftarrow (-C_0^{-1})\ \underline{e};\ \ \underline{u}^{(1,1)} \leftarrow \underline{u}^{(1)}\ ; \qquad (3.12)$$

$$\underline{v}^{(1)} \leftarrow 2(-C_0^{-1})\ \underline{u}^{(1)};\ \underline{v}^{(1,1)} \leftarrow \underline{v}^{(1)}\ ; \qquad (3.18)$$

$$G^{(1)}(0) \leftarrow (-C_0^{-1})\ \Lambda^{(0)};\ G^{(1,1)}(0) \leftarrow G^{(1)}(0); \qquad (3.11)$$

$$U^{(1)} \leftarrow (-C_0^{-1})^2\ \Lambda^{(0)};\ U^{(1,1)} \leftarrow U^{(1)}\ \} \qquad (3.15)$$

B1.2 Levels 2 to N.

for n = 2 **to** N **do**

$$\{\ (-C_{n-1}^{-1}) \leftarrow -(A^{(n-1)} + M^{(n-1)}(-C_{n-2}^{-1})\ \Lambda^{(n-2)})^{-1}\ ; \qquad (2.3)$$

$$\underline{u}^{(n)} \leftarrow (-C_{n-1}^{-1})(\underline{e} + M^{(n-1)}\underline{u}^{(n-1)}\ ; \qquad (3.13)$$

$$\underline{u}^{(1,n)} \leftarrow \underline{u}^{(1,n-1)} + G^{(1,n-1)}(0)\ \underline{u}^{(n)}\ ; \qquad (3.14)$$

$$\underline{v}^{(n)} \leftarrow 2(-C_{n-1}^{-1})(I + M^{(n-1)}U^{(n-1)})\ \underline{u}^{(n)}$$

$$+ (-C_{n-1}^{-1})\ M^{(n-1)}\underline{v}^{(n-1)}; \qquad (3.19)$$

$$\underline{v}^{(1,n)} \leftarrow \underline{v}^{(1,n-1)} + 2\, U^{(1,n-1)}\underline{u}^{(n)} + G^{(1,n-1)}(0)\, \underline{v}^{(n)}; \qquad (3.20)$$

$$G^{(n)}(0) \leftarrow (-C_{n-1}^{-1})\, \Lambda^{(n-1)}; \qquad (3.11)$$

$$G^{(1,n)}(0) \leftarrow G^{(1,n-1)}(0)\, G^{(n)}(0); \qquad (3.8)$$

$$U^{(n)} \leftarrow (-C_{n-1}^{-1})(I + M^{(n-1)}U^{(n-1)})(-C_{n-1}^{-1})\, \Lambda^{(n-1)}; \qquad (3.16)$$

$$U^{(1,n)} \leftarrow U^{(1,n-1)}G^{(n)}(0) G^{(1,n-1)}(0)\, U^{(n)} \} \qquad (3.17)$$

B2 Stationary distribution.

B2.1 Initialization of the recurrence.

$$\{ C_N \leftarrow A^{(N)} + M^{(N)}(-C_{N-1}^{-1})\, \Lambda^{(N-1)}; \qquad (2.3)$$

$$\underline{\pi}_N \leftarrow \text{solution of } \underline{\pi}_N C_N = \underline{0}\ ,\ \underline{\pi}_N \underline{e} = 1 \} \qquad (2.5)$$

B2.2 Determination of $\underline{P}_n$, $0 \leqslant n \leqslant N$.

<u>for</u> n = N-1 <u>to</u> 0 <u>do</u>

$$\{ \underline{\pi}_n \leftarrow \underline{\pi}_{n+1} M^{(n+1)}(-C_n^{-1}); \qquad (2.6)$$

(re-normalize)

$$\alpha \leftarrow \sum_{k=n}^{N} \underline{\pi}_k \underline{e} ;$$

<u>for</u> k = n <u>to</u> N <u>do</u>

$$\{ \underline{\pi}_k \leftarrow \alpha^{-1} \underline{\pi}_k \} \quad \}$$

TABLES

Table I. Cumulative probabilities-number of machines on repair.

Table II. Marginal distribution-number of prey under surveillance. Missing numbers are less than 5.10^{-6}.

Table III. Values of γ giving the same reduction as doubling the number of predators.

[illegible]	0	1	2	3	4	5
$a(\lambda_0)$	.56395	.84593	.95872	.99256	.99932	1
[illegible]	.56395	.84593	.95872	.99256	.99932	1
[illegible]	.56398	.84589	.95870	.99255	.99932	1
[illegible]	.56421	.84561	.95850	.99249	.99931	1
[illegible]	.56579	.84397	.95711	.99202	.99925	1
[illegible]	.56910	.84151	.95424	.99091	.99908	1
[illegible]	.57033	.84078	.95321	.99047	.99900	1
[illegible]	.57050	.84068	.95306	.99040	.99899	1
[illegible]	.49516	.79226	.93486	.98620	.99852	1
[illegible]	.72118	.93754	.98946	.99881	.99993	1

Table I

Cumulative probabilities-number of machines on repair

n	I	II	III	IV	V
0	.00073		.00048		
1	.00820		.00543	.00005	.00003
2	.04019		.02662	.00061	.00043
3	.11319		.07504	.00428	.00342
4	.20375	.00002	.13524	.01959	.01724
5	.24653	.00026	.16400	.06175	.05835
6	.20544	.00220	.13776	.13712	.13601
7	.11886	.01270	.08407	.21606	.22012
8	.04755	.05082	.04948	.24021	.24628
9	.01294	.13835	.05546	.18523	.18752
10	.00233	.24903	.08481	.09610	.09441
11	.00026	.28299	.09427	.03198	.03001
12	.00002	.18866	.06255	.00633	.00560
13		.06531	.02160	.00066	.00054
14		.00933	.00308	.00003	.00002
15		.00031	.00010		

Table II

Limiting distribution-number of prey under surveillance.
Missing numbers are less than 5.10^{-6}

m	$t_{5,1}$	$t_{10,1}$	$t_{15,1}$	$t_{5,2}$	$t_{10,2}$	$t_{15,2}$
15	4.1	4.4	7.8	4.9	7.1	> 15
20	4.1	4.0	3.9	4.5	5.6	5.4

Table III

Values of γ giving the same reduction as doubling the number of predators

FIGURES

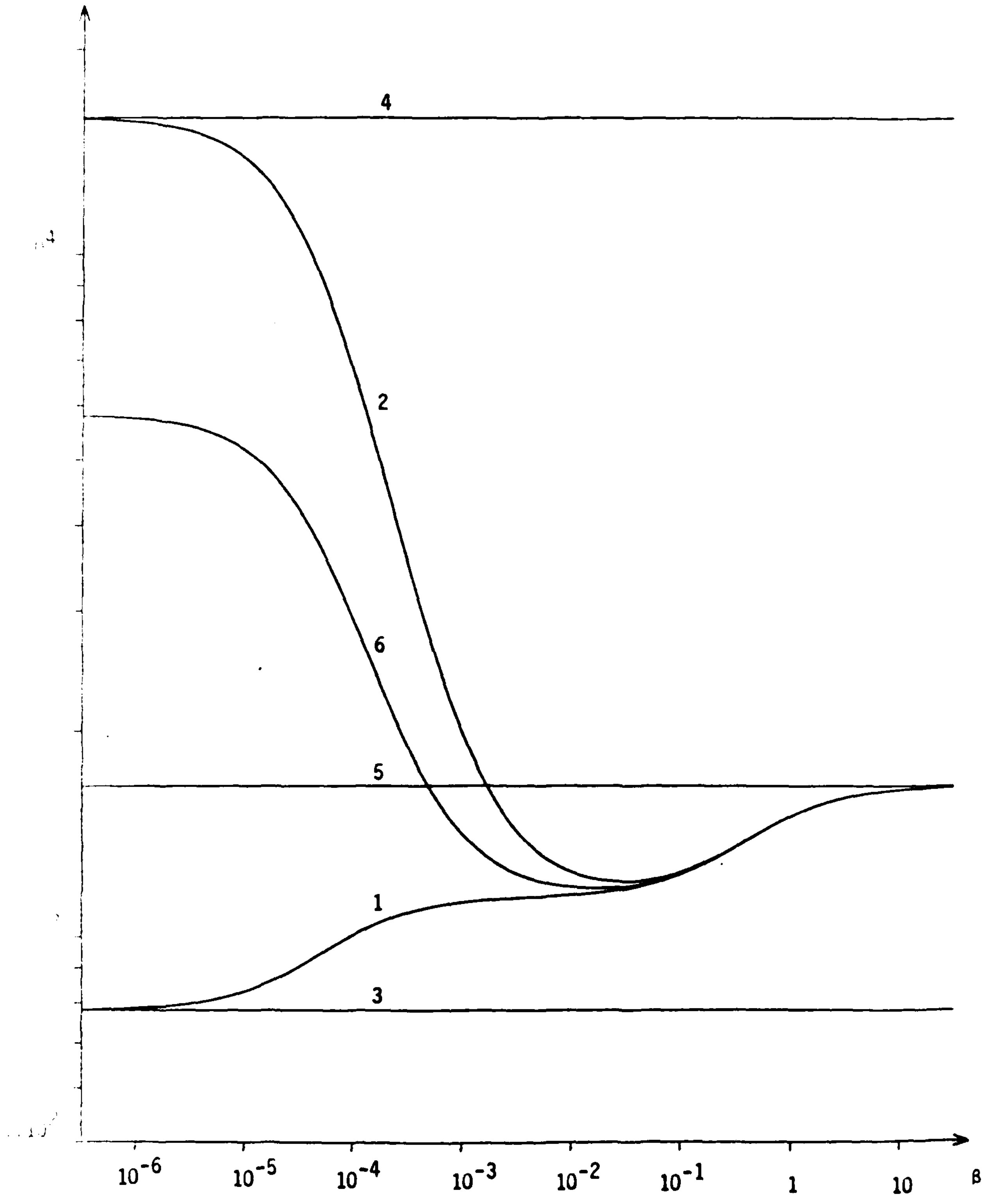

Fig.1. Expected time to reach the state [all machines on repair], starting from [no machine on repair].

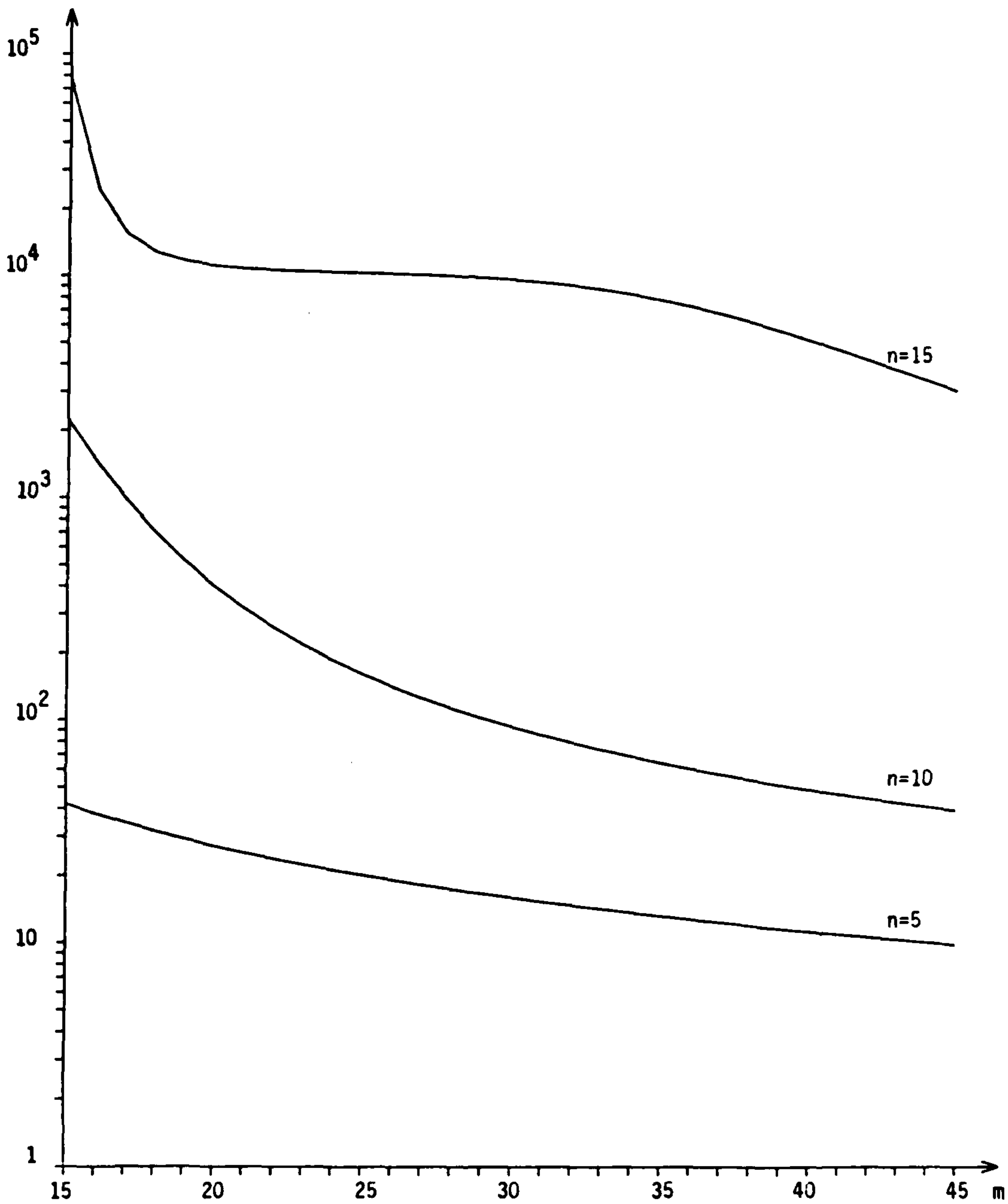

Fig.2. Expected time to reach the states [n prey under observation], starting from [0 prey under observation, environment 1], varying number of predators.

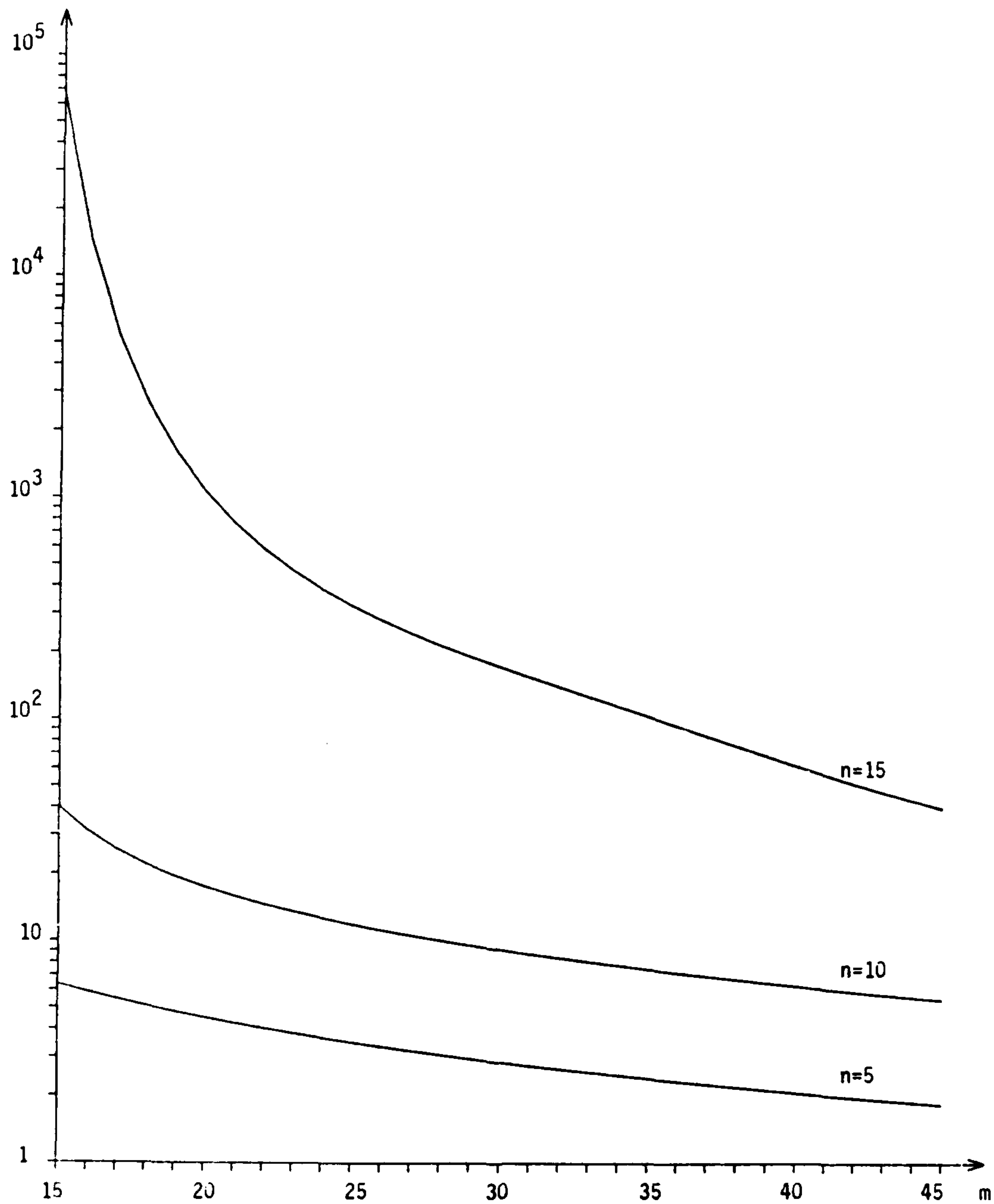

Fig.3. Expected time to reach the states [n prey under observation], starting from [0 prey under observation, environment 2], varying number of predators.

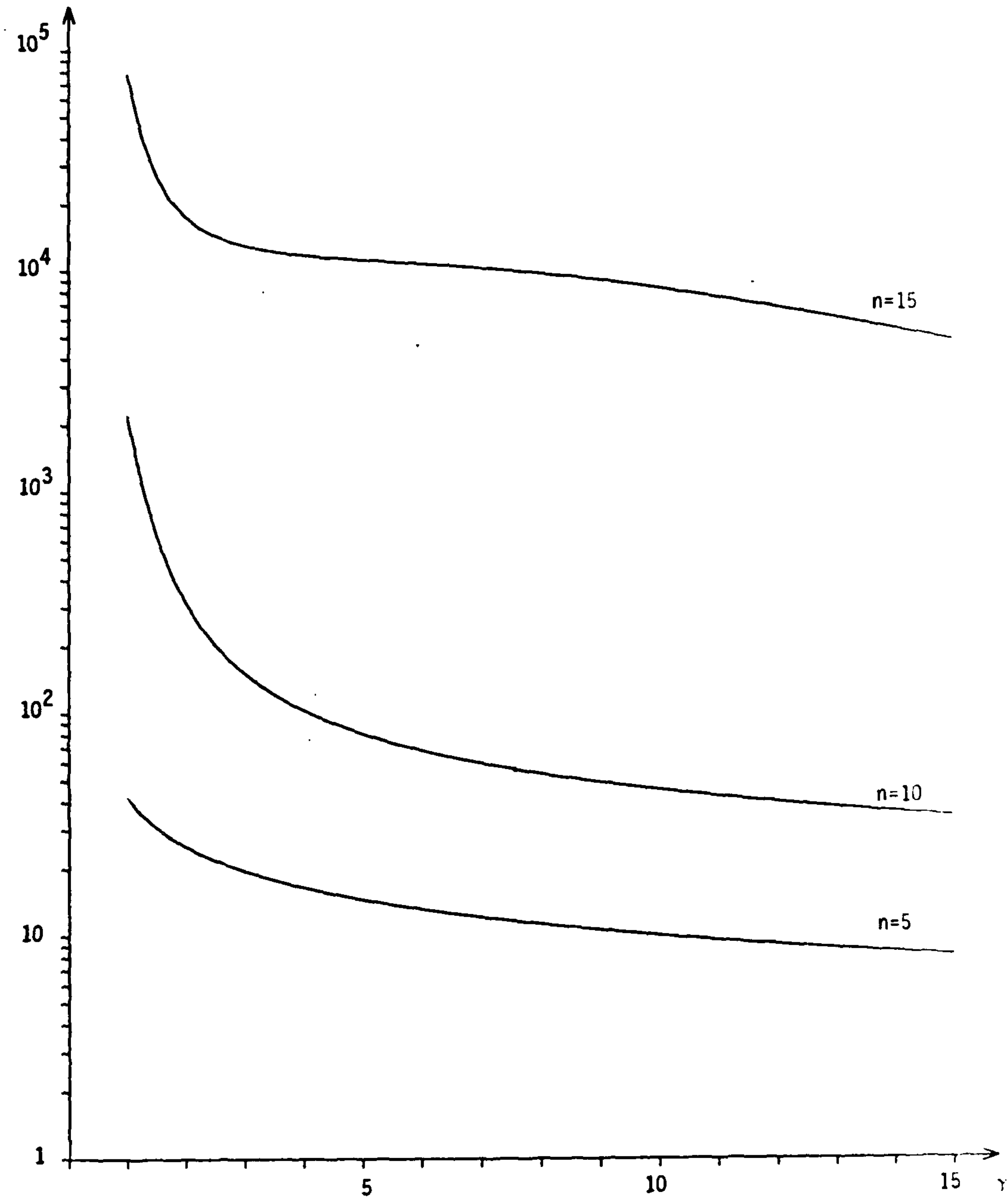

Fig.4. Expected time to reach the states [n prey under observation], starting from [0 prey under observation, environment 1], 15 predators, varying probability ratio.

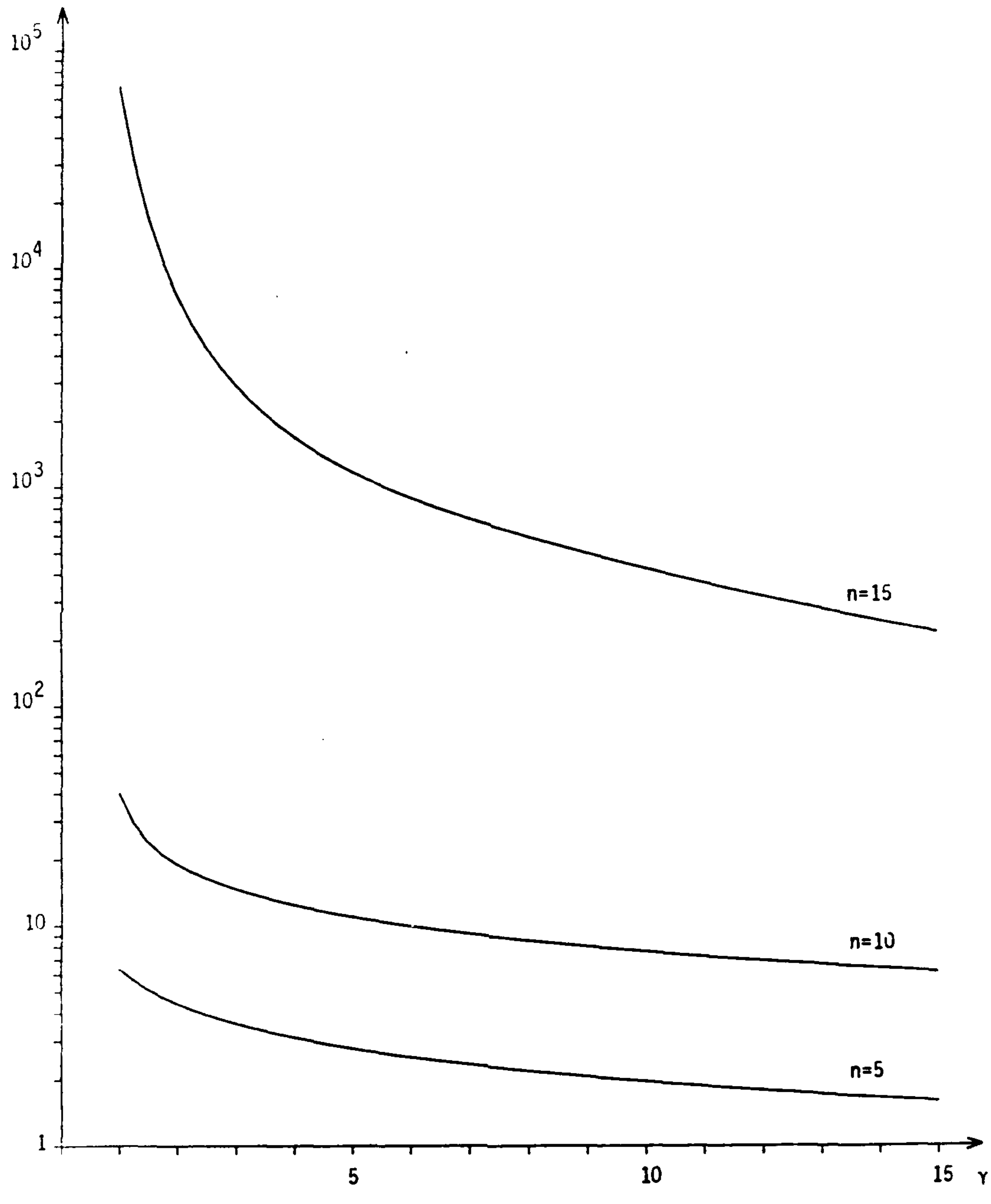

Fig.5. Expected time to reach the states [n prey under observation], starting from [0 prey under observation, environment 2], 15 predators, varying probability ratio.

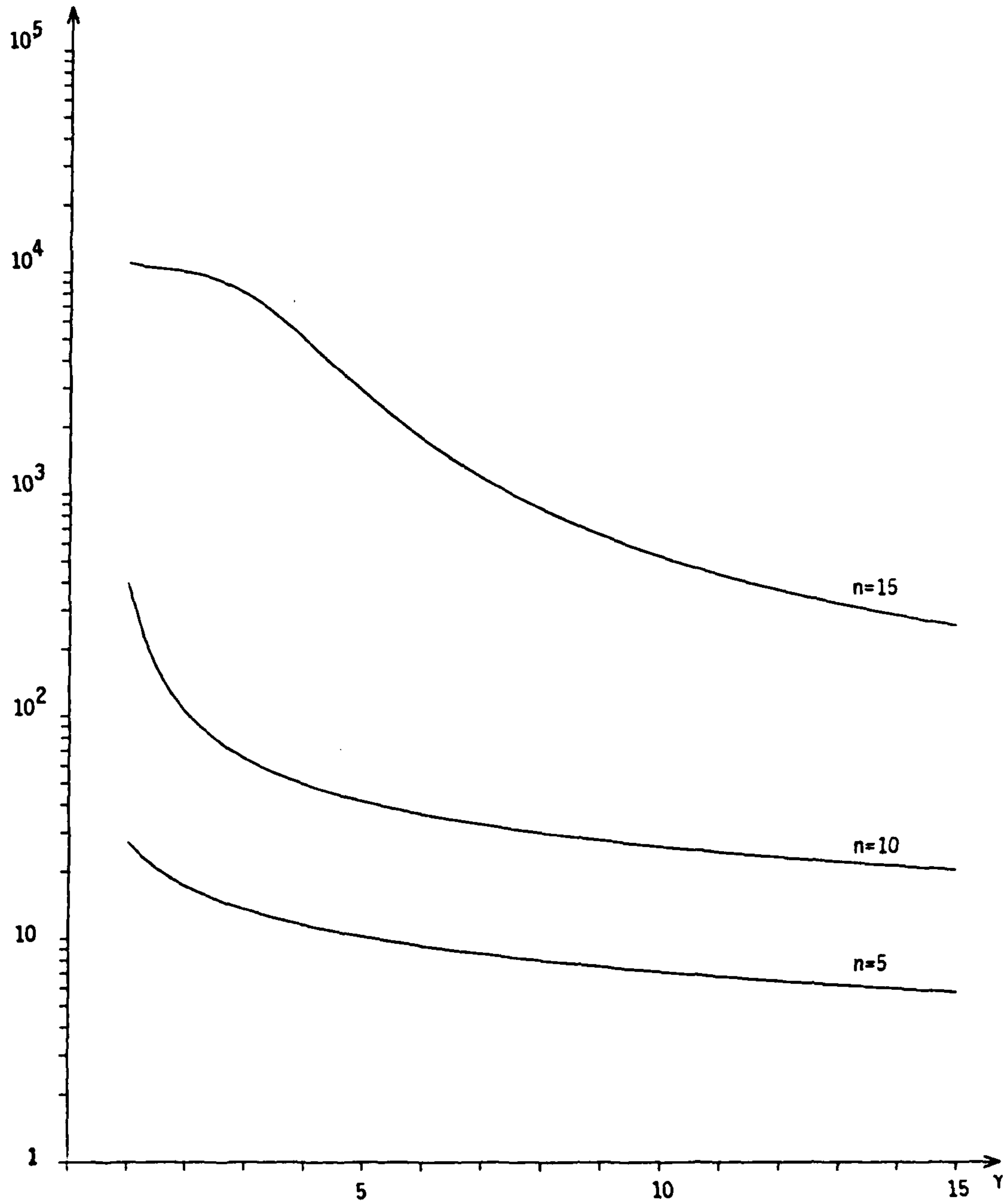

Fig.6. Expected time to reach the states [n prey under observation], starting from [0 prey under observation, environment 1], 20 predators, varying probability ratio.

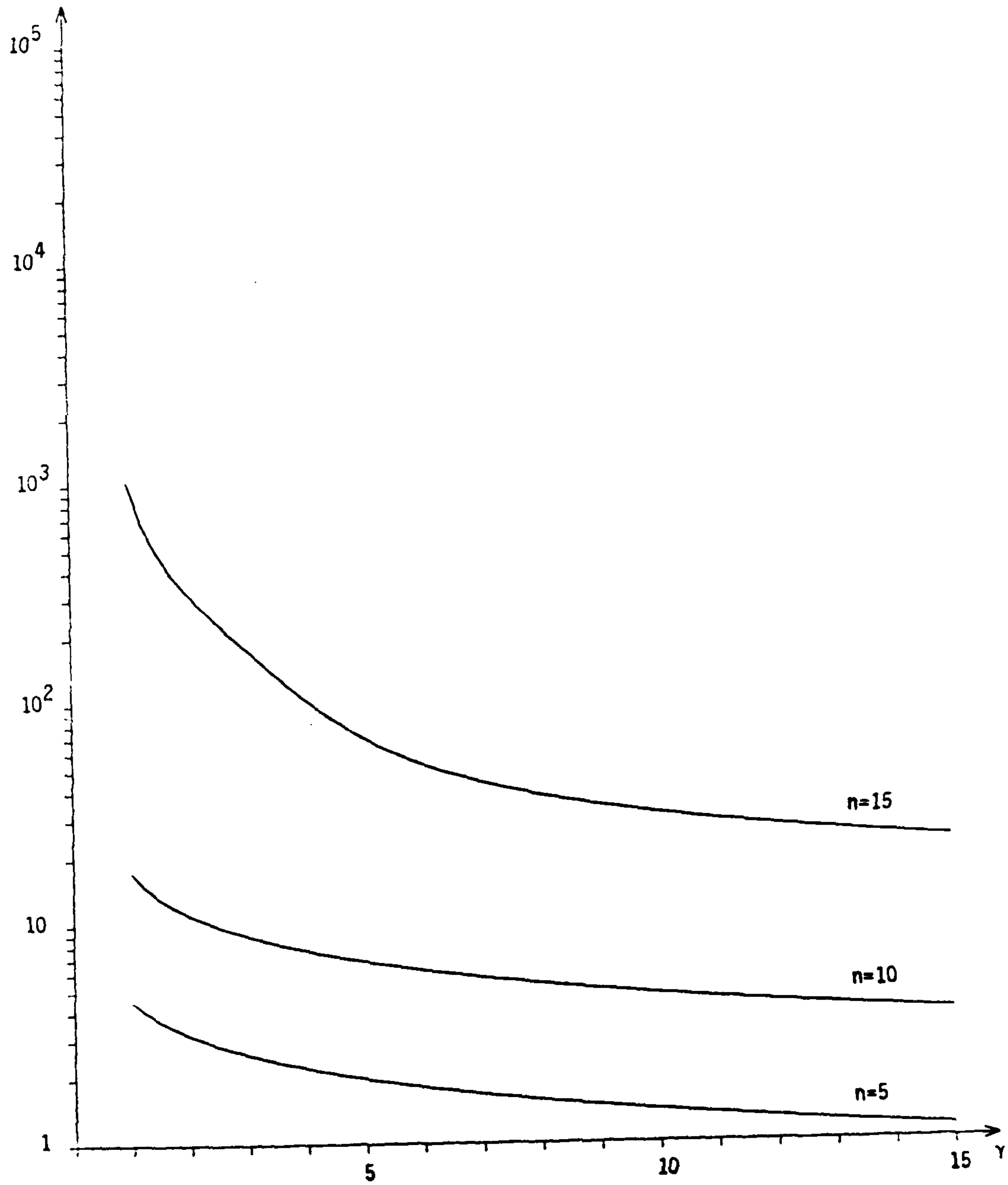

Fig.7. Expected time to reach the states [n prey under observation], starting from [0 prey under observation, environment 2], 20 predators, varying probability ratio.

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